Readers' Forum

Brief discussions of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A Discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on "Time Finite Element Discretization of Hamilton's Law of Varying Action"

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IN Ref. 1 Hamilton's Law is used as a variational source for the derivation of finite element discretization procedures in the time domain. The writers of this Comment share with those authors the opinions that "the extension of the finite element to the time domain is well motivated" and it "bears attractive properties of much greater accuracy than other existing stable methods, and easy computer implementation," but the algorithms proposed in Ref. 1 somewhat miss the stated objectives and they perpetuate the artificial and tricky presentation of a previous paper by the same authors.[2]

The purpose of this Comment is to show how the artificiality of the formulation is due only to an incorrect interpretation of the Hamilton's Law and how it should be removed. In fact, if the authors are so strongly convinced that "the lowest order interpolation set is the third-order Hermitian polynomials," there is no need to complicate the matter by writing Hamilton's Law as in Eq. (4) of Ref. 1 and then using for the variation function an expression like that of Eq. (8).

Once the Hermitian polynomials are taken as approximation, a simple and straightforward way to obtain substantially the same results as Ref. 1 is shown in Ref. 3. Moreover Ref. 4, starting from the formulation of Ref. 3, attains a step-by-step intergration formula, paralleling Eqs. (A3) and (A4) of Ref. 1 and possessing the same stability bound. Furthermore in Ref. 4 one can also find the same presentation of "The Modified Algorithm with Unconditional Stability," as that presented in the Appendix of Ref. 1.

The authors of Ref. 1 are not fully exploiting the power of Hamilton's Law when it is written in the variational form of Eq. (4). This power comes from the fact that the approximating functions have to ensure continuity only of the displacement, and not of any of its derivatives, because of the presence of the boundary terms, without losing the capability to correctly enforce any set of independent initial and/or boundary conditions.

This is simply demonstrated by approximating q with a linear interpolation, i.e.:

$$q(t)=\left(1-\frac{t-t_0}{\Delta t}\right)q_0+\frac{t-t_0}{\Delta t}q_f \tag{1a}$$

$$\dot{q}(t)=\frac{1}{\Delta t}(q_f-q_0) \tag{1b}$$

with $\Delta t=t_f-t_0$, and assuming for S the same interpolating functions of q, i.e.:

$$S(t)=\left(1-\frac{t-t_0}{\Delta t}\right)S_0+\frac{t-t_0}{\Delta t}S_f \tag{2a}$$

$$\dot{S}(t)=(1/\Delta t)(S_f-S_0) \tag{2b}$$

It is now important to understand that, in order to correctly apply Hamilton's Law, the boundary term should not be approximated in any way with Eq. (1b), but must be written simply as it stands, i.e.:

$$-S^TM\dot{q}\Big|_{t_0}^{t_f}=-S_fM\dot{q}_f+S_0M\dot{q}_0 \tag{3}$$

Substituting Eqs. (1-3) into Eq. (4) of Ref. 1, we obtain:

$$\left(\frac{1}{\Delta t}M+\frac{1}{2}C+\frac{\Delta t}{6}K\right)q_f-\left(\frac{1}{\Delta t}M+\frac{1}{2}C-\frac{\Delta t}{3}K\right)q_0$$
$$=M\dot{q}_0+\int_{t_0}^{t_f}\left(1-\frac{t-t_0}{\Delta t}\right)f\mathrm{d}t \tag{4a}$$

$$-\left(\frac{1}{\Delta t}M-\frac{1}{2}C-\frac{\Delta t}{3}K\right)q_f+\left(\frac{1}{\Delta t}M-\frac{1}{2}C+\frac{\Delta t}{6}K\right)q_0$$
$$=-M\dot{q}_f+\int_{t_0}^{t_f}\frac{t-t_0}{\Delta t}f\mathrm{d}t \tag{4b}$$

Given q_0 and $\dot{q}_0$, Eq. (4a) can be used to compute q_f, after which $\dot{q}_f$ is easily obtained from Eq. (4b), thus producing a step-by-step integration formula. Otherwise, Eqs. (4a) and (4b) can be assembled to obtain a whole solution over a time of interest. In the latter case, during the assembly procedure the $M\dot{q}$ terms from two contiguous time intervals can either cancel each other or be equal to an applied external impulse, but in any case the correct requirement of continuity or jump of velocity is satisfied in a natural way. The assembly process is in fact unnecessary, as it gives rise to a triangular system whose solution traces down the step-by-step integration procedure.

Finally, it is important to note that, using approximating functions that give a discontinuity in velocity and a null acceleration, we have obtained a step-by-step integration formula that maintains the symmetry of the matrices, does not double the order of the system, and can take into account impulsive forces, ensuring the correct variation of momentum. None of these features is present in the development of Ref. 4, even if it started from the same basic equations.

Clearly higher order approximations can be profitably used and the reader is referred to Ref. 5 for more details on the use of Eq. (4) of Ref. 1 in dynamic response problems. Reference 5 also contains a discussion of the stability and precision properties of the step-by-step integration formulas obtainable with the application of the previously presented concepts.

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[1]Riff, R. and Baruch, M., "Time Finite Element Discretization of Hamilton's Law of Varying Action," *AIAA Journal*, Vol. 22, Sept. 1984, pp. 1310-1318.

[2]Baruch, M. and Riff, R., "Hamilton Principle, Hamilton's Law, 6^n Correct Formulations," *AIAA Journal*, Vol. 20, May 1982, pp. 687-692.

[3]Fried, I., "Finite Element Analysis of Time Dependent Phenomena," *AIAA Journal*, Vol. 7, June 1969, pp. 1170-1173.

[4]Geradin, M., "A Classification and Discussion of Integration Operators for Transient Structural Response," AIAA Paper 74-105, 1974.

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Reply by Authors to M. Borri, M. Lanz, and P. Mantegazza

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We would like to thank Professor Borri, Lanz, and Mantegazza for their interest in our paper[1] and for sharing with us the opinion that the extension of the finite element to the time domain is well motivated. It seems, however, that the authors of the Comment missed the main goal of the paper,[1] which was to introduce a new high-precision time finite element analogous to the standard finite element algorithms for which cases it may be needed.

The authors[1-3] (see also Refs. 4 and 5) are still convinced that the easiest and most consistent way to impose all of the initial dynamic conditions and to produce more accurate algorithms is through the third or higher order Hermitian polynomials. It appears,[2-3] and was pointed out in the paper,[1] that the so-called "straightforward way" in the Comment, which was introduced in Ref. 4, caused the time finite element algorithm to become unconditionally unstable.[2] On the contrary, the so-called "artificial and tricky" formulation of the paper,[1,6] succeeded in producing a stable converged time finite algorithm of sixth-order accuracy.

Moreover, in spite of the same stability characteristic, the step-by-step version of the algorithm presented in the Appendix of Ref. 1 possesses a much greater accuracy (by two orders) than the one presented in Ref. 5, which is based on the algorithm of Ref. 4. Furthermore, in spite of the same kind of modification (see Ref. 7) for unconditional stability, the algorithm presented in the Appendix of Ref. 1 is of fifth-order accuracy, while the one of Ref. 5 is of third-order accuracy due to the difference in the basic algorithms.

Of course, one can choose to exploit the power of time finite element discretization of Hamilton's Law to reproduce any already existing step-by-step integration methods. Clearly Eqs. (4) of the Comment are nothing else than the well known Newmark linear acceleration method of third-order accuracy.

In fact, the first author of this Reply had previously obtained[3] the same result as in the Comment and has introduced a general formulation of this kind of step-by-step method by applying time finite element discretization of Hamilton's Law. Similar results had been obtained earlier by Zienkiewicz[8] through weighted residuals methods.

The general formulation[3] is simply supported by approximating the displacement q and the variation S as follows,

$$q(t) = (1-\tau)q_{i-1} + \tau q_i \tag{1}$$

$$S(t) = G_i(\tau)S_{i-1} + G_2(\tau)S_i \tag{2}$$

where $\tau = t/\Delta t_i$. By substitution of Eqs. (1) and (2) into Eq. (4) of Ref. 1 (Hamilton's Law), one obtains:

$$\begin{bmatrix} -M+\gamma_1\Delta t_i C+(\beta_1-\gamma_1)\Delta t_i^2 K, & M-\gamma_1\Delta t_i C-\beta_1\Delta t_i^2 K \\ M-\gamma_2\Delta t_i C-(\beta_2-\gamma_2)\Delta t_i^2 K, & -M+\gamma_2\Delta t_i C+\beta_2\Delta t_i^2 K \end{bmatrix} \times \begin{Bmatrix} q_{i-1} \\ q_i \end{Bmatrix} = \Delta t_i^2 \begin{bmatrix} \beta_1-\gamma_1, & -\beta_1 \\ -\beta_2+\gamma_2, & \beta_2 \end{bmatrix} \begin{Bmatrix} f_{i-1} \\ f_i \end{Bmatrix} \tag{3}$$

where,

$$\beta_\alpha = \int_0^1 G_\alpha \tau d\tau \Big/ \int_0^1 \dot{G}_\alpha \tau d\tau \quad \gamma_\alpha = \int_0^1 G_\alpha d\tau \Big/ \int_0^1 \dot{G}_\alpha d\tau; \ \alpha = 1,2 \tag{4}$$

Clearly, many known and new procedures can be developed assuming different values for β_α and γ_α. For example,

$$\gamma_1 = -\frac{1}{2};\ \gamma_2 = \frac{1}{2};\ \beta_1 = -\frac{1}{6};\ \beta_2 = \frac{1}{3} \tag{5}$$

will produce the linear acceleration method obtained in the Comment, and the reader is referred to Ref. 3 for more details.

However, the authors believe that one does not fully exploit the power of the method by reproducing well-known existing operators, as for example in the Comment and these were not the stated objectives of the paper.[1] The purpose of the paper was to show that the finite element method based on the variational statement of Hamilton's Law[6] and applied in the time domain is capable of systematic derivation of many new highly accurate procedures for the solution of initial value problems.

References

[1]Riff, R. and Baruch, M., "Time Finite Element Discretization of Hamilton's Law of Varying Action," *AIAA Journal*, Vol. 22, Sept. 1984, pp. 1310-1318.

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